Close Thu: 15.5
Closing Tue: Taylor Notes 1, 2
Close next Thu: Taylor Notes 3

Motivation "the see-saw"

15.5 Center of Mass

New App: Given a thin uniformly distributed plate (a *lamina*) with density at each point $\rho(x, y)$ can we find the center of mass (*centroid*).

$\rho(x, y)$ = mass/area (kg/m^2)

We will see that

$$\bar{x} = \frac{\text{Moment about y}}{\text{Total Mass}} = \frac{\iint_R x\, p(x,y)dA}{\iint_R p(x,y)dA}$$

$$\bar{y} = \frac{\text{Moment about x}}{\text{Total Mass}} = \frac{\iint_R y\, p(x,y)dA}{\iint_R p(x,y)dA}$$

In general:

If you are given ***n* points**

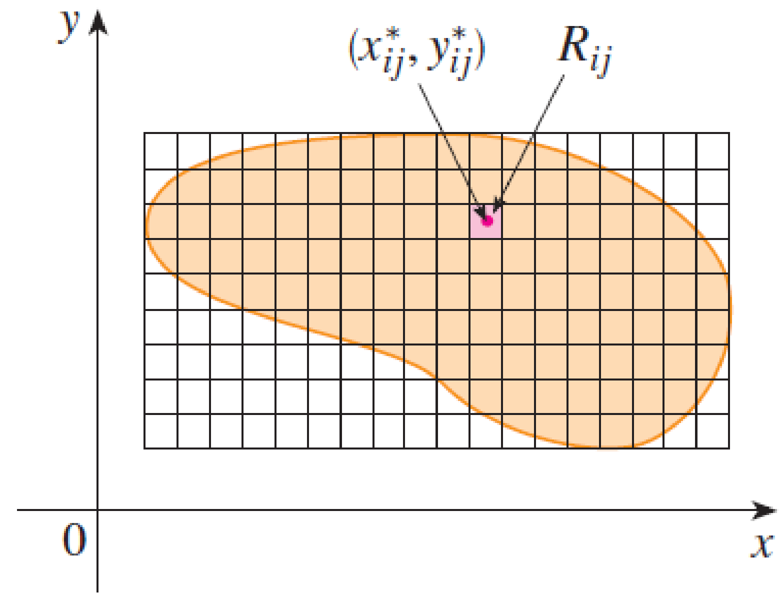
$(x_1,y_1), (x_2,y_2), \ldots, (x_n,y_n)$ with

corresponding masses $m_1, m_2, \ldots, m_n$

then

$$\bar{x} = \frac{m_1x_1 + \cdots + m_nx_n}{m_1 + \cdots + m_n} = \frac{M_y}{M}$$

$$\bar{y} = \frac{m_1y_1 + \cdots + m_ny_n}{m_1 + \cdots + m_n} = \frac{M_x}{M}$$



Derivation:

1. Break region into m rows and n columns.
2. Find center of mass of each rectangle:
$$(\bar{x}_{ij}, \bar{y}_{ij})$$
3. Estimate the mass of each rectangle:
$$m_{ij} = p(\bar{x}_{ij}, \bar{y}_{ij})\Delta A$$
4. Now use the formula for *n* points.
5. Take the limit.

$$\bar{x} = \frac{\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}x_{ij}}{\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}}$$
$$= \frac{\sum_{i=1}^{m}\sum_{j=1}^{n} \bar{x}_{ij}\, p(\bar{x}_{ij}, \bar{y}_{ij})\Delta A}{\sum_{i=1}^{m}\sum_{j=1}^{n} p(\bar{x}_{ij}, \bar{y}_{ij})\Delta A}$$

Center of Mass:

$$\bar{x} = \frac{\text{Moment about y}}{\text{Total Mass}} = \frac{\iint_R \; x\, p(x,y)dA}{\iint_R \; p(x,y)dA}$$

$$\bar{y} = \frac{\text{Moment about x}}{\text{Total Mass}} = \frac{\iint_R \; y\, p(x,y)dA}{\iint_R \; p(x,y)dA}$$

Example:

Consider a 1 by 1 m square metal plate. The density is given by $p(x,y) = kx$ kg/m^2 for some constant k.

Find the center of mass.

Side note: The density $p(x,y) = kx$ means that the density is proportional to x which can be thought of as the distance from the y-axis. In other words, the plate gets heavier at a constant rate from left-to-right.

Translations:

Density proportional to the dist. from…

…the y-axis -- $p(x,y) = kx.$

…the x-axis -- $p(x,y) = ky.$

…the origin -- $p(x,y) = k\sqrt{x^2 + y^2}.$

Density proportional to the <u>square</u> of the distance from the origin:

$$p(x,y) = k(x^2 + y^2).$$

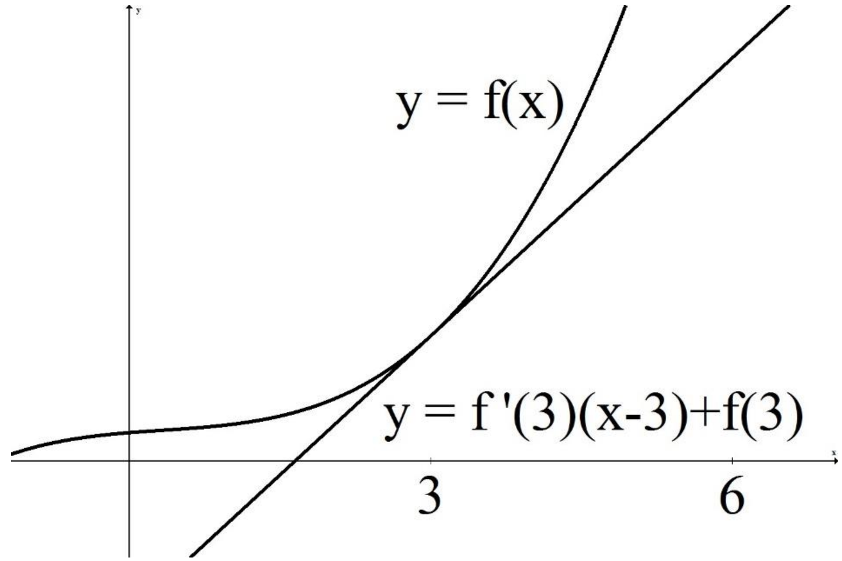
Density <u>inversely</u> proportional to the distance from the origin:

$$p(x,y) = \frac{k}{\sqrt{x^2 + y^2}}$$

Example: A thin plate is in the shape of the region bounded between the circles of radius 1 and 2 in the first quadrant.
The density is proportional to the distance from the origin.
Find the center of mass.

Taylor Notes 1 (TN 1):
Tangent Line Error Bounds

Goal: Approximate functions with tangent lines and get error bounds. And begin a process of better and better approximations.



Def'n:
We say the **first Taylor polynomial for f(x) based at b** (or the tangent line approximation) is

$$T_1(x) = f(b) + f'(b)(x-b)$$

Note: For x close to b

$$f(x) \approx T_1(x)$$

The "error" between then would be

$$\text{Error} = |f(x) - T_1(x)|$$

Warm up: An upper **bound**, M, for a function is a number that is always bigger than that function.

Examples: Find an upper **bound** for the function on the given interval:

1. $|\sin(5x)|$ on $[0,2\pi]$

2. $|x-3|$ on $[1,5]$

3. $\left|\frac{1}{(2-x)^3}\right|$ on $[-1,1]$

4. $|\sin(x) + \cos(x)|$ on [0,2π]

5. $\left|\cos(2x) + e^{-2x} + \frac{6}{x}\right|$ on [1,4]

Tangent Linear Error Bound Thm
(1st case of Taylor's Inequality)
If |f''(x)| ≤ M for all *x* on [*a*,*b*], then

$$|f(x) - T_1(x)| \leq \frac{M}{2}|x - b|^2.$$

for all x on [*a*,*b*].

Note:

M = some upper bound on f''(x)

$|x - b|$ =distance x is away from b.

Proof sketch for x > b:

Start with $f(x) - f(b) = \int_b^x f'(t)dt$.

By parts with u = f'(t), dv = dt gives

$$f(x) - f(b) = f'(b)(x - b) - \int_b^x (t - x)f''(t)dt$$

so

$$f(x) - f(b) - f'(b)(x - b) = \int_b^x (x - t)f''(t)dt.$$

Thus,

$$|f(x) - T_1(x)| = \left|\int_b^x (x - t)f''(t)dt\right|$$

Then note

$$\left|\int_b^x (x - t)f''(t)dt\right| \leq \int_b^x (x - t)|f''(t)|dt$$

$$\leq M\int_b^x (x - t)dt$$

$$= \frac{M}{2}(x - b)^2$$

To use the Tangent Line Error Bound:

1. Find f’’(t).
2. Find an upper bound for |f’’(t)|. Call this M.
3. Use the theorem.
4. Plug in x = “an endpoint” to get a single number for the largest upper bound.

Two types of error bound questions in the current homework:

A) Given interval, find error bound.
B) Given error bound, find interval.

Example:

Let f(x) = ln(x).

1. Find the 1st Taylor polynomial based at b=1.
2. Find a bound on the error over the interval

   J = [1/2, 3/2]
3. Find an interval around b = 1 where the error is less than 0.01.